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An Episode in the History of Dynamics: Jakob Hermann's Proof (1716–1717) of Proposition 1, Book 1, of Newton's *Principia*

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Proposition 1 of Book 1 of Newton's *Principia* (1687), which states that Kepler's area law holds for any central force, plays a fundamental role in the study of central force motion. Newton's geometric proof of this proposition is based on an intuitive theory of limits. In 1716–1717 the Swiss mathematician, Jakob Hermann, gave a proof of Proposition 1 based on infinitesimals. The present paper discusses both Newton's and Hermann's solutions. A comparison of the two gives us an insight into an episode of the process that led from the geometric style of Newton's *Principia* to the analytic style of Euler's *Mechanica* (1736). © 1996 Academic Press, Inc.

Lehrsatz 1 aus dem ersten Buch von Newtons *Principia* (1687) spielt eine wesentliche Rolle in der Behandlung der Zentralkräften. Er sagt, daß für eine beliebige Zentralkraft das zweite Keplersche Gesetz gilt. Newton gab einen geometrischen Beweis dieses Lehrsatzes, der sich auf einer anschaulichen Theorie der Grenzen stützt. 1716–1717 gab der schweizerische Mathematiker Jakob Hermann einen Beweis des Lehrsatzes 1, der sich auf der Benutzung der infinitesimalen Grössen stützt. In diesem Artikel werden Newtons und Hermanns Beweise erörtert. EinVergleich zwischen den beiden Beweisen erlaubt ein Verständnis eines Moments der geschichtlichen Entwicklung, die von der geometrischen Denkweise Newtons *Principia* zu der analytischen Denkweise Eulers *Mechanica* (1736) geht. © 1996 Academic Press, Inc.

La Proposizione 1 del Libro 1 dei *Principia* (1687) di Newton gioca un ruolo fondamentale nello studio delle forze centrali. In questa Proposizione si afferma che la legge delle aree di Kepler vale per qualsiasi forza centrale. Newton ne diede una dimostrazione geometrica basata su una teoria intuitiva dei limiti. Nel 1716–1717 il matematico svizzero Jakob Hermann ne diede una dimostrazione basata sugli infinitesimi. In questo articolo vengono presentate la soluzione di Newton e quella di Hermann. Un confronto fra le due ci permette di apprezzare un episodio del processo che ha portato dallo stile geometrico dei *Principia* di Newton allo stile analitico della *Mechanica* (1736) di Euler. © 1996 Academic Press, Inc.

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The title of Isaac Newton's *magnum opus*, *Philosophiae naturalis principia mathematica*, hides a puzzle for historians of science. Any student who has read about "Newtonian dynamics" might think that the "mathematical principles" to which Newton refers are those of infinitesimal calculus. Newton is, in fact, rightly remembered as one of the discoverers of calculus, the "method of series and fluxions" as he named it. However, as historians of dynamics know too well, in the *Principia* calculus is used in only a few isolated cases: the rules for differentiating a product

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or a power are stated, and even power series are employed. In other cases, the calculus is referred to in a rather allusive way: in fact, in some propositions Newton states that the problem is solved by ''granting the quadrature of the figures,'' i.e., he assumes the existence of a method for squaring the curvilinear area subtended by a curve.[1] By and large, Newton prefers to adhere to geometrical demonstrations. Very often in his geometry limits of ratios and limits of sums, as well as infinitesimals by various orders, occur. A ''translation'' into the language of calculus seems therefore implicitly suggested.

The following passage from the preface of L'Hospital's *Analyse des infiniment petits* remains famous:

> Furthermore, it is a justice due to the learned M. Newton, and that M. Leibniz himself accorded to him: That he has also found something similar to the differential calculus, as it appears in his excellent book entitled *Philosophiae Naturalis Principia Mathematica*, published in 1687, which is almost entirely about this calculus. [19, ii^{v}–iii^{r}][2]

Any attempt to describe the mathematical methods employed in the *Principia* has to take into consideration the plurality, complex stratification, and peculiarity of Newton's geometrical procedures. It would be simplistic to talk about *the mathematical method* of Newton's *Principia*. Geometry is, in fact, employed in Newton's dynamics for a variety of purposes, including propositions (e.g., 1, 2, and 7–13 in Book 1) where geometry is used to estimate the limit of ratios and sums of ''vanishing'' geometrical quantities: propositions (e.g., 39–42 in Book 1) where the study of the relationships between infinitesimal geometrical quantities allows one to reduce a dynamical problem to the quadrature of a curve; propositions (e.g., 66 in Book 1) where geometry suggests the study of the qualitative behaviour of a dynamical system. If these diverse geometrical procedures were simply calculus in disguised form, as L'Hospital, or rather his ghostwriter Fontenelle, seems to maintain, translation into the language of Leibniz's differential or Newton's fluxional calculi would be a *routine* exercise. This, however, was not the case. The mathematicians who, at the beginning of the 18th century, set themselves the task of applying the calculus to Newton's dynamics (most notably Pierre Varignon, Jakob Hermann, and Johann Bernoulli) had to surmount difficult problems. In some cases the goal was achieved, but quite often recourse to geometry was still necessary.

Newton himself was well aware of the distance that separated the calculus of fluxions from the geometrical methods employed in the *Principia*. In fact, when, in the first decade of the 18th century, the dispute with Leibniz broke out over who was first to invent the calculus, he was not able to make much use of the *Principia* as proof of his knowledge of the algorithm of fluxions. He could only

[1] See, for instance, Propositions 39 to 42 in Newton's *Principia* [23, 125–134].

[2] Translation by D. T. Whiteside in [29, 447]. In what follows, for Newton's *Principia,* I use the standard Motte and Cajori edition [23]. If not indicated, translations are mine. ''C'est encore une justice dûë au sçavant M. Newton, & que M. Leibniz luy a renduë luy-même: Qu'il avoit aussi trouvé quelque chose de semblable au Calcul différentiel, comme il paroît par l'excellent Livre intitulé *Philosophiae Naturalis Principia Mathematica*, qu'il nous donna en 1687: lequel est presque tout de ce calcul.''

refer to a few propositions [29, 442ff]. The bulk of the work, he had to admit, was "demonstrated synthetically" [29, 598–599].[3] He sought to maintain that he had discovered "most of the propositions" with the help of the "new analysis," but that it was now difficult to see the analysis utilized in the process of discovery. He further suggested an analogy between his mathematical procedures and those of the "Ancients." In 1714, speaking of himself in the third person, he wrote:

> By the help of this new analysis Mr. Newton found out most of the propositions in his Principia Philosophiae. But because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Propositions synthetically that the systeme of the heavens might be founded upon good Geometry. And this makes it now difficult for unskillful men to see the Analysis by which those Propositions were found out. [29, 598–599]

However, external and internal evidence is against Newton's statement. The preparatory manuscripts of the *Principia* reveal no use of calculus and seem to indicate that Newton wrote them in the form in which they were published [see, e.g., 28]. An internal analysis of the structure of the demonstrations in the *Principia* furthermore reveals that Newton's geometrical dynamics is, in many significant cases at least, to a certain extent independent of calculus techniques.[4] This independence should be neither overstressed nor overlooked. Defining Newton's mathematical methodology in dynamics is thus a complex task, not only because of the plurality of Newton's geometrical techniques, but also because of their contiguity with the calculus. In some cases, the geometrical demonstrations of the *Principia* lend themselves to an almost immediate translation into calculus concepts; in other cases this translation is complicated, unnatural, or even problematic. Needless to say, notwithstanding Newton's rhetorical declaration of continuity between his methods and the methods of the "Ancients," his geometrical dynamics is a wholly 17th-century affair.

Today, we take it for granted that calculus is a better suited tool than geometry for dealing with dynamics. But at the beginning of the 18th century, the choice of mathematical methods to be applied to dynamics was problematic. First of all, a *plurality* of geometrical methods had to be compared not with *a* calculus, but with a *plurality* of calculi. Calculus came in at least two forms (the differential and the fluxional) and could be based either on infinitesimal concepts or on limits. Second, the way in which dynamical concepts should be represented was not (and is not) obvious. Thirdly, the calculus (at least up to Euler) was never thought of as completely independent from geometrical representation.[5] Mathematicians of the early 18th century debated such issues.

[3] Newton, trying to find a bridge with the calculus of fluxions, defined the geometrical methods of the *Principia* as a "methodus synthetica fluxionum et momentorum" ("a synthetical method of fluxions and moments") [29, 454–455].

[4] Proposition 1 is an example.

[5] To give just an example: differential equations could be manipulated as symbolical expressions, but retained their meaning as equations among differential quantities, quantities which could be geometrically represented.

It is sometimes argued that this debate favoured a process of degeometrization of dynamics, a process which goes straight from the geometry of the *Principia* to the analytics of Euler or Lagrange. Furthermore, several historians draw a sharp distinction between a British geometrical and a Continental analytical school: the British followers of Newton are thus depicted as conservative geometricians who were superseded by progressive algebraists. There is more than some truth to this picture, and this paper will give some support to the idea. However, the subject of this paper is very ''local'' (an ''episode'' as the title says), and I warn the reader against generalizations. It is a fact that in Britain analytical methods in dynamics were employed (by Newton himself!), while on the Continent even convinced Leibnizians sometimes deployed geometry.

The complexity of the process of mathematization of dynamics is particularly evident in transitional figures, such as Jakob Hermann (1678–1733). He belonged to the Basel school headed by the Bernoullis and, sharing their methodology, made important contributions to the analytical treatment of dynamics. However, he also leaned towards Newton and, on many occasions, preferred to deal with dynamical problems in terms of geometry. His methodology is thus quite eclectic. Hermann was a pupil of Jakob I Bernoulli in Basel and was a remote relative of Leonhard Euler. He belonged to the group of brilliant Swiss mathematicians who contributed to the early development of Leibniz's calculus. Leibniz supported his career in various ways. Thanks to Leibniz's recommendation, Hermann held chairs of mathematics in Padua from 1707 to 1713 and in Frankfurt-an-der-Oder until 1724, while from 1724 to 1731 he was connected with the Academy in St. Petersburg. He was able to return home only in 1731 when he took up the professorship of ethics and natural law, the chair of mathematics being occupied by Johann Bernoulli.[6]

Hermann's main work is *Phoronomia*, which, written during his Italian period, was published in Amsterdam in 1716 [17]. This work is devoted to the dynamics of solid and fluid bodies and covers many problems dealt with by Newton in the first two books of the *Principia*. In the preface, Hermann declares his intention of adhering to geometrical methods, since these seem to him more suitable for beginners [17, vii–viii]. However, his knowledge of calculus is evident in the way in which he deals with infinitesimals. Hermann's *Phoronomia* is indeed representative of the process of transition that transformed dynamics in the first decades of the 18th century.

The process was concluded only in the late thirties by Leonhard Euler, when the Swiss mathematician was able to offer a general, uniform, and well-regulated analytical method for approaching the dynamics of the *Principia*. Euler in his *Mechanica* (1736) wrote:

> But what pertains to all the works composed without analysis, is particularly true with mechanics. In fact, the reader, even though he is persuaded about the truth of the things that are demonstrated, cannot nonetheless understand them clearly and distinctly. So that he is hardly able

[6] For information on Hermann's life and work, see [11; 20; 22].

> to solve with his own strengths the same problems, when they are changed just a little, if he does not inspect them with the help of analysis and if he does not develop the same propositions into the analytical method. This is exactly what happened to me, when I began to study in detail Newton's *Principia* and Hermann's *Phoronomia*. In fact, even though I thought that I could understand the solutions to numerous problems well enough, I could not solve problems that were slightly different. Therefore I strove, as much as I could, to get at the analysis behind those synthetical methods in order to deal with, for my own purposes, those propositions in terms of analysis. Thanks to this procedure I perceived a remarkable improvement of my knowledge. [9, 8][7]

Thus, Euler openly contrasted his general analytical method with both the perplexing complexity of Newton's geometrical procedures in the *Principia* and the interplay between calculus and geometry employed by Hermann in his *Phoronomia.*

In this paper, I will focus on an episode, a significant one I believe, of the history leading from the Baroque complexity of the *Principia* to the Enlightenment of Euler's *Mechanica,* viz., Hermann's proof of the very first proposition of the *Principia*.

Proposition 1, Book 1 plays a fundamental role in the dynamics of central forces. It says that if a body P is accelerated by a central force directed towards or away from a fixed centre S, then Kepler's area law holds, i.e., the radius vector from S to P sweeps equal areas in equal times. Proposition 1 is important since, for central force motion, it identifies a constant of motion: areal velocity (in modern terms, angular momentum). Furthermore, since the area swept by SP is proportional to time, Newton could use this area as a geometric representation of time. In Proposition 2, Newton showed the inverse of Proposition 1. Proposition 2 says that if the area law holds (i.e., if there is a point S in the plane of the trajectory, and SP sweeps equal areas in equal times), then P is accelerated by a central force directed towards or away from S.

It is interesting to see how, in a modern textbook [13, 557–558], these two propositions are proved. The most natural choice is to use polar coordinates (r, θ), so that the origin coincides with the centre of force. The radial and transverse accelerations are thus expressed by the following two formulae:

$$a_r = d^2r/dt^2 - r\,(d\theta/dt)^2, \text{ and} \tag{1}$$

$$a_t = rd^2\theta/dt^2 + 2dr/dt\, d\theta/dt. \tag{2}$$

Let A be the area swept out by the radius vector (therefore $2\, dA/dt = r^2\, d\theta/dt$ and $2d^2A/dt^2 = r^2d^2\theta/dt^2 + 2r\, dr/dt\, d\theta/dt = ra_t$). For a central force, a_t is equal

[7] "Sed quod omnibus scriptis, quae sine analysi sunt composita, id potissimum Mechanicis obtingit, ut Lector, etiamsi de veritate earum, quae proferuntur, convincatur, tamen non satis claram et distinctam eorum cognitionem assequatur, ita ut easdem quaestiones, si tantillum immutentur, proprio marte vix resolvere valeat, nisi ipse in analysin inquirat easdemque propositiones analytica methodo evolvat. Idem omnino mihi, cum Neutoni *Principia* et Hermanni *Phoronomiam* perlustrare coepissem, usu venit, ut, quamvis plurium problematum solutiones satis percepisse mihi viderer, tamen parum tantum discrepantia problemata resolvere non potuerim. Illo igitur iam tempore, quantum potui, conatus sum analysin ex synthetica illa methodo elicere easdemque propositiones ad meam utilitatem analytice pertractare, quo negotio insigne cognitionis meae augmentum percepi."

to zero. It follows that, by integration of (2), $dA/dt = k$ (i.e., the areal velocity is equal to a constant k). Inversely, if $dA/dt = k$, by differentiation, it follows that a_t is zero (i.e., the force is central). The double implication of Newton's Propositions 1 and 2 is thus embedded in (2).

The above demonstration is quite straightforward: mathematically speaking, it requires only elementary calculus and the use of polar coordinates. Equations (1) and (2) were only worked out in the 1740s, despite the fact that elementary calculus and polar coordinates were already in use in the late 17th century. Bertoloni Meli, in his essay [4], has given abundant evidence that the first expressions of (1) and (2) are to be found in the works of Daniel Bernoulli [2], Euler [10], and Alexis-Claude Clairaut [6] carried out in the 1740s. In their studies on constrained motion (typically a ball in a rotating tube) and planetary motions, these mathematicians arrived at expressions for radial and transverse acceleration. Once this representation is achieved, the proof of Propositions 1 and 2 is simple. As we shall see, this representation was not available to Hermann, who resolved acceleration into a tangential and a normal component. The fact that the mathematical means was there,[8] but its application to dynamics was lacking, teaches us something about the complexities of the history of mathematical dynamics. Progress comes not only from the discovery of new mathematics but also from the understanding of how mathematics can be applied to dynamical concepts.

Newton gave a geometric proof of Proposition 1. In the *Principia* we read:

> Proposition I. The areas which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described. [23, 40]

Newton's demonstration has been the object of several interesting studies, most notably E. Aiton [1] and D. T. Whiteside [28, 121–122], both of whom have raised objections relative to the correctness of Newton's limit procedures.[9] Furthermore, R. S. Westfall [27, 411–413] has noticed that the concept of force utilized by Newton is open to ambiguities. I will not discuss these subtle aspects.

Newton's proof is as follows. Divide the time into equal and finite intervals, Δt_1, Δt_2, Δt_3, etc. At the end of each interval the "centripetal force acts at once with a great impulse" [23, 40], and the velocity of the body changes instantaneously. The resulting trajectory (see Fig. 1) is a polygonal ABCDEF. The areas SAB, SBC, SCD, etc. are swept by the radius vector in equal times. By applying the first two laws of motion, it is possible to show that they are equal. In fact, if at the end of Δt_1, when the body is at B, the centripetal force did not act, the body would continue in a straight line with uniform velocity (because of the first law of motion). This means that the body would reach c at the end of Δt_2, so that AB = Bc. Triangles SAB and SBc have equal areas. However, we know that at the end of Δt_1, when the body is at B, the centripetal force acts. Where is the body at the end of Δt_2?

[8] At the beginning of the 18th century, polar coordinates were widely used to represent trajectories.
[9] For a critical evaluation of Whiteside's and Aiton's theses, see Erlichson's paper [8]. See also Fraser's paper [12].

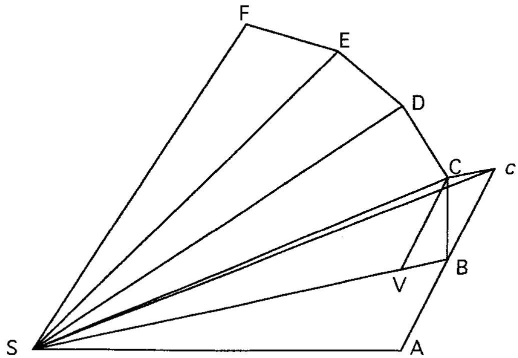

FIG. 1. Diagram for central force motion adapted from Newton's *Principia*, Proposition 1.

In order to answer this question, one has to consider how Newton, in Corollary 1 to the laws, defines the mode of action of two forces acting "simultaneously": "A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately" [23, 14]. Invoking the above corollary, Newton deduces that the body will move along the diagonal of parallelogram BcCV, reaching C at the end of Δt_2. Cc is parallel to VB, therefore triangles SBc and SBC have equal areas. Triangles SAB and SBC thus have equal areas. One can iterate this reasoning and construct points C, D, E, F. They all lie in a plane, since the force is directed towards S, and the areas of triangles SCD, SDE, SEF, etc. are equal to the area of triangle SAB. The body therefore describes a polygonal trajectory which lies on a plane, and the radius vector SP sweeps equal areas SAB, SBC, SCD, etc., in equal times. Newton passes from the polygonal to the smooth trajectory with a limit argument based on the method of prime and ultimate ratios.[10] He writes:

> Now let the number of those triangles be augmented, and their breadth diminished in infinitum: and ... their ultimate perimeter ADF will be a curved line: and therefore the centripetal force, by which the body is continually drawn back from the tangent of this curve, will act continually: and any described areas SADS, SAFS, which are always proportional to the times of description, will, in this case also, be proportional to those times. [23, 41]

That is to say, since Kepler's area law always holds for any discrete model (polygonal trajectory generated by an impulsive force), and since the continuous model (smooth trajectory generated by a continuous force) is the limit of the discrete models for $\Delta t \to 0$, then the area law holds for the continuous model. The area swept by SP is proportional to time.

In Hermann's *Phoronomia*, central forces are treated in Chapter 2, Section 2 of Book 1, entitled "On the curvilinear motions in void under any hypothesis on the

[10] The method of prime and ultimate ratios was developed by Newton in Section 1, Book 1 of the *Principia*. It consists of a geometric foundational theory for limit procedures.

variation of gravity."[11] The motion of a mass point accelerated by a central force is one of the main topics of the first book of the *Principia.* Hermann had achieved important results in this field. For instance, in 1710 he had published in the Paris *Mémoires* a solution [16], based on differential calculus, of the so-called inverse problem of central forces (given an inverse square central force and initial condition of a mass point, determine the trajectory).[12] His solution was published together with those of Johann Bernoulli [3] and Pierre Varignon [26]. Newton in the first edition of the *Principia* (Corollary 1 to Propositions 11–13, Book 1) had just stated, without offering a proof, that conic sections are the only solutions to the inverse problem of central forces [23, 61]. The solutions of Hermann, Bernoulli, and Varignon could thus be seen as a victory of the Leibnizian calculus over the geometry of prime and ultimate ratios deployed by Newton.[13] However, the three Continental mathematicians relied on the law of areas; i.e., they equated the infinitesimal element of time dt and the infinitesimal area dA swept by the radius vector, so that

$$dA = kdt, \tag{3}$$

where k is a constant. The only proof then available that, for any central force, (3) holds was that given by Newton in the *Principia*. As Newton's proof was markedly geometrical in character, Bernoulli's, Hermann's, and Varignon's analytical solutions of the inverse problem of central forces ultimately relied on the geometry of the *Principia*.

Newton's geometric demonstration of the area law (3) was assumed in the treatment of central forces until Hermann, in his *Phoronomia* [17, 69–71], gave a proof of it in terms of differentials. The same proof was restated, in a notation more accessible to a modern reader, in a "letter" to John Keill that Hermann published in 1717 in the *Journal litéraire* [18, 411–415]. The difference basically involves the fact that in the *Phoronomia*, Hermann expresses the differentials by referring to geometric points of a figure. For instance, if s is the arclength and p the perpendicular from force centre to tangent, the respective differentials are denoted in the *Phoronomia* with letters like Nn and Cc, while in the *Journal litéraire* they appear as ds and dp (see Fig. 2). This, of course, is not a negligible difference.

Hermann proves Newton's Proposition 1 as follows [17, 69–71; 18, 411–415]. First of all, he *assumes* that the trajectory will be a plane curve ANB (see Fig. 2). Hermann introduces the following geometrical constructions and symbols (the symbols occur only in the 1717 paper, and I will define them in parentheses). The force centre is D; Nn (= ds) is the infinitesimal element of arc, NC is the tangent in N, nc the tangent in n. The two tangents meet in e. DC (= p) is perpendicular to the tangent NC and meets it in C, while it cuts the tangent nc in c. ON and On are perpendicular to the tangents NC and nc, so that O is the centre of the osculating

[11] "De motibus curvilineis in vacuo, in quacunque gravitatis variabilis hypothesi."

[12] Hermann's solution of the inverse problem has been recently analysed by Silvia Mazzone in [21].

[13] On this topic, see [5; 14; 15; 24; 30].

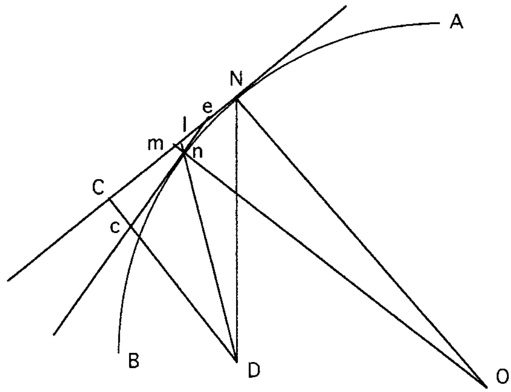

FIG. 2. Diagram for central force motion adapted from Hermann [16; 17]. Symbols: ON = ρ, DC = p, NC = q, DN = r, Nn = ds, Cc = dp, nl = $d\alpha$, nm = $d\beta$. Symbols occur only in [17].

circle (therefore ON = ρ). Finally, let nl (= $d\alpha$) and nm (= $d\beta$) be two infinitesimal segments meeting the tangent NC in l and m. There is an ambiguity about the inclination of nl and nm. In one representation they are the prolongation of Dn and On, respectively. However, in another representation, nl is parallel to DN, and nm parallel to DC. This ambiguity is typical of infinitesimal techniques. The difference between the two representations can be ignored since higher order infinitesimals are cancelled.[14] Other symbols will be defined when they occur.

Hermann resolves the central force F into two components, F_N normal to the trajectory and F_T tangent to the trajectory. He shows that

$$F_N \rho = v^2 \text{ and} \tag{4}$$

$$F_T\, ds = -v\, dv, \tag{5}$$

where a point of unit mass is considered, v is the velocity, s is the arclength, and ρ is the radius of curvature. Equation (4) means that the resultant force normal to the trajectory is equal to the square of the speed divided by the radius of curvature. Equation (5) means that the force tangent to the trajectory is equal to the rate of change of the speed.

It is important to consider how Hermann deduced Eqs. (4) and (5). Actually, both were part of the standard *repertoire* of early 18th-century mathematicians. For instance, both could be found in Newton's *Principia*: Eq. (4) as Proposition 4 and Eq. (5) as Proposition 40 of Book 1. Hermann proceeds as follows. He begins with two "general principles" valid for a uniform force G (a "pesanteur uniforme")

[14] Notice also that the centre of the osculating circle is determined by the intersection of normals to two neighbouring points; ds is considered a straight segment, while Cc is assumed equal to dp, higher order infinitesimals being neglected.

which accelerates from rest a body of unit mass in rectilinear accelerated motion. The "first general principle already known to Newton and Varignon" [18, 412][15] is

$$Gt = v, \tag{6}$$

where v is the velocity, and t the time. The second general principle is

$$(2l/G)^{1/2} = t, \tag{7}$$

which gives the time of fall from rest after the distance l is covered. These two principles are applied to the curvilinear motion caused by the centripetal force F. Hermann states that, during the infinitesimal interval of time (= dt) required to traverse the infinitesimal arc Nn (= ds), the force F can be assumed constant (in both modulus and direction). He goes on: "The tangential force is precisely that which causes the varied movement along the curve ANB" [18, 413].[16] That is, the rate of change of speed is due to F_T. Therefore, the first principle (6) applied to the infinitesimal increment of speed dv acquired after dt yields

$$F_T\, dt = -dv, \tag{8}$$

and (5) therefore follows (since $F_T\, v\, dt = -v\, dv$ and $v\, dt = ds$).

If during dt the body is accelerated by a constant force (which varies in neither modulus nor direction), nl (= $d\alpha$) may be conceived, according to Hermann, as a small Galileian fall, and from the second principle (7)

$$2\, d\alpha/F = dt^2 = ds^2/v^2. \tag{9}$$

Therefore, from (9),

$$d\alpha = (ds^2 F)/(2v^2). \tag{10}$$

Furthermore,

$$\mathrm{nm}/\mathrm{Nn} = \mathrm{Nn}/2\mathrm{ON}, \tag{11}$$

or, in symbols,

$$d\beta = ds^2/2\rho. \tag{12}$$

Equation (11) (and its equivalent (12)) need some clarification. Since Hermann cancels third-order infinitesimals, he identifies the trajectory at N with the osculating

[15] "C'est là notre prémier principe general, qui à déja été donné par M. Newton & par M. Varignon il y a long temps."

[16] "[La force tangentielle] est précisement celle qui cause le mouvement varié sur la courbe ANB."

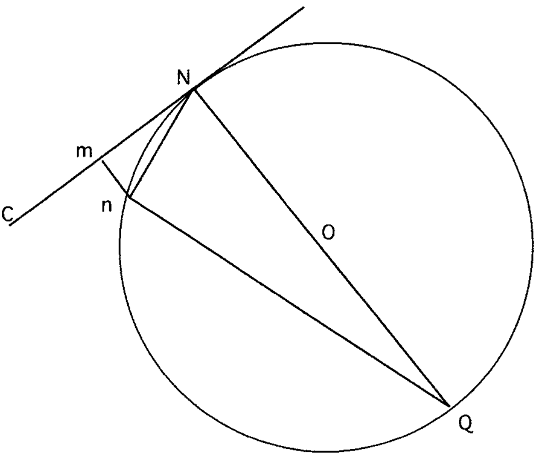

FIG. 3. Osculating circle at N of the trajectory ANB. Triangle Nmn is similar to triangle QnN.

circle at N. The versed sine nm is normal to the tangent NC; therefore triangles Nmn and QnN are similar (see Fig. 3). And (11) follows.

Now reconsider Fig. 2. Since (by construction) $F/F_N = d\alpha/d\beta$:

$$F/F_N = d\alpha/d\beta = (ds^2F)/(2v^2)\,(2\rho/ds^2) = F\rho/v^2, \tag{13}$$

and equation (4) follows.

Now consider DN ($= r$), NC ($= q$), and DC ($= p$):

$$F_T/F_N = q/p. \tag{14}$$

Furthermore, from the similarity of triangles Cec and NOn (and canceling higher-order infinitesimals),

$$dp/q = ds/\rho, \tag{15}$$

where Cc ($= dp$) and Nn ($= ds$).[17]

Hermann divides (5) by (4) and gets

$$dv/v = -(F_T\,ds)/(F_N\,\rho). \tag{16}$$

The ratio $(F_T\,ds)/(F_N\,\rho)$, because of (14) and (15), is equal to dp/p. Therefore,

$$dv/v = -dp/p. \tag{17}$$

At last, integrating (17), Hermann arrives at

$$pv = 2k, \tag{18}$$

[17] The reader might want to prove (15) from $ds/dr = r/q$ and the expression for the radius of curvature $\rho = rdr/dp$.

which is equivalent to (3), since $pv = pds/dt = 2dA/dt$ (pds is twice the area DNn $= dA$ swept by the radius vector). Hermann in the *Phoronomia* observes that this is what the "illustrious Newton for the first time proved in Prop. 1, Book 1 of *Principia* but from a completely different ground" [17, 71].[18]

Newton's and Hermann's demonstrations that for any central force Kepler's area law holds are indeed quite different.[19] While Newton relies on an intuitive geometrical limit procedure, Hermann develops his proof in terms of differential equations of motion.

The recourse to a limiting procedure gives a rigour to Newton's argument that is reminiscent of Archimedean demonstrations by "exhaustion." While Newton made rhetorical use of the similarities of his techniques with those of the "Ancient Geometers" in order to defend the logical correctness of the procedures employed in the *Principia*, it should be pointed out that his geometrical dynamics is extremely innovative and cannot be identified with "classical" techniques. In Proposition 1, geometry is applied to dynamical concepts in a way typical of 17th-century natural philosophy (Christiaan Huygens, rather than Archimedes, can be taken as a precursor of Newton from this point of view). Newton's limit procedure (which allows a transition from the discrete to the continuous model of the trajectory) could be conceived only by a mathematician acquainted with 17th-century infinitesimal techniques and with the kinematical geometry introduced by, e.g., Gilles Personne de Roberval and Isaac Barrow. However, it is true that Newton's geometrical demonstration of Proposition 1 is independent of the calculus of fluxions. In fact, Newton's demonstration can be understood by a mathematician who knows nothing about fluxional or differential algorithms. The difficulties—and disagreements—that modern interpreters have had in translating Newton's proof in terms of modern analytics are a sign of the distance between the geometry of the *Principia*'s Proposition 1 and the analytical techniques of calculus. Newton's demonstration of Proposition 1 is thus integrated into the scheme of 17th-century geometrical dynamics but does not belong to the conceptual scheme of calculus.

Hermann's demonstration, in contrast, is integrated into the conceptual scheme of Leibnizian calculus from the very beginning. The trajectory is represented locally in terms of differentials (ds, dp, $d\alpha$, $d\beta$, dv). Finite quantities, such as F_T, F_N, and ρ, are then expressed in terms of ratios of differentials. The study of the geometrical and dynamical relationships of infinitesimals (see Eqs. (6)–(15)) leads to differential equations (see Eqs. (4) and (5)) which can be manipulated algebraically until, thanks to an integration, the result sought is achieved (see Eqs. (16)–(18)). The geometry of infinitesimals is thus the model from which one can work out differential equations.

Hermann's procedure, not Newton's, will be the standard of mid-18th-century

[18] "Illustris Newtonus id primus demonstravit Prop. I Lib. I *Princ. Phil. Nat. Math.* sed ex diversissimo fundamento."

[19] It should be noted that a comparison of Hermann's and Newton's demonstrations is made difficult by the fact that they use not only different mathematical methods but also different mathematizations of force (most notably, in Proposition 1, Newton does not resolve force into a tangential and a normal component).

dynamics. In fact, the interplay between the geometry of infinitesimals and differential equations characterizes much of Euler's work on dynamics. Euler, from this point of view, in the 1730s follows procedures similar to those employed by Hermann in proving Proposition 1. Hermann's demonstration of 1716–1717 was, in fact, included in Euler's *Mechanica* [9, 194–195] of 1736. Euler, however, in the *Mechanica* criticized Hermann and (as we have seen in the quotation above) put him in Newton's camp. In his *Phoronomia*, Hermann had not followed the same method systematically, but rather had displayed a variety of methodologies. Euler, on the other hand, was trying in the *Mechanica* to come up with a general method to be applied to dynamics. Furthermore, a decade later Euler (e.g., in [10]) avoided the resolution of acceleration into tangential and radial components and preferred to tackle central forces with equations such as (1) and (2) (see [4, 312ff]). In this new conceptual context, the relationship between central forces and Kepler's area law can be proved in a more radically analytical fashion (see our discussion of Eqs. (1) and (2) above).

It is interesting to note that Hermann criticizes John Keill in the *Journal litéraire* because the Scottish mathematician in his treatment of central forces relies on "the analogies of the common principle of proportionality that exists between times and areas" and stresses that he has been the first to give a proof of this principle [18, 411].[20] Notwithstanding his admiration for Newton, Hermann seems to imply that Proposition 1 was not really proved in the *Principia* and that reference to it could just be an "analogy." Immediately after the *Principia*'s publication, the status of the law of areas was not so definite as one might think. For instance, in the Leibnizian *milieu*, one can find Varignon, who, in 1700, dealt with central forces assuming different laws for the time of orbital motion. He wrote:

> My first purpose was to find the central forces of the planets according to the hypothesis of Kepler, Newton, and Leibniz, as it is more physical, making $dt = rrd\theta$; but after having considered that this hypothesis is not the unique one adopted in astronomy, here is how I can deal with all the hypotheses ... whatever might be the hypothesis on the times, or the values of dt. [25, 225][21]

Varignon then proceeds to consider cases in which the time is proportional to an angle measured from an equant point. Notice that both Hermann and Varignon read the *Principia* carefully, which was for them an important point of departure. Nonetheless, it seems that they considered Proposition 1 unconvincing as proved by Newton. Thus, Hermann could refer to his 1716 demonstration as the first true proof, free from dubious analogies, of the law of areas. It has been justly stressed

[20] "Il [Keill] tire son Theoréme à force d'analogies du commun principe de la proportion qu'il y a entre les temps & les aires respectives de la courbe; au lieu que dans la route que j'ai prise, le premier à ce que je sache."

[21] "... mon premier dessein étoit de ne chercher les forces centrales des Planetes que dans l'hypothese de Kepler, de M. Newton. & de M. Leibnitz, comme la plus physique, en me proposant de faire par tout $dt = r\,dz$ [$dz = r\,d\theta$]; mais ayant depuis fait refléxion que cette hypothese des temps n'est pourtant pas la seule qui se fasse en Astronomie, voici comment je satisfais à toutes ... quelques soient d'ailleurs ces hypotheses de temps, ou les valeurs de dt."

that early 18th-century mathematicians, particularly on the Continent, accepted calculus mostly on pragmatic grounds (see, for instance, [7, 264–265]). Notwithstanding the lack of clarity at a foundational level, calculus techniques proved to be a more efficient and productive tool compared with the more rigourous geometrical techniques, a fact which accounts for their acceptance. However, at least in some cases, the contrary seems to have happened. A demonstration given in terms of differential equations was seen by Hermann as superior to the one given in terms of geometry. The study of these cases can help us appreciate how, in the 18th century, dynamics was gradually transformed into an analytic discipline.

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